A skeleton approximate solution of the Einstein field equations for multiple black-hole systems

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An approximate analytical and non-linear solution of the Einstein field equations is derived for a system of multiple non-rotating black holes. The associated space-time has the same asymptotic structure as the Brill-Lindquist initial data solution for multiple black holes. The system admits an Arnowitt-Deser-Misner (ADM) Hamiltonian that can particularly evolve the Brill-Lindquist solution over finite time intervals. The gravitational field of this model may properly be referred to as a *skeleton* approximate solution of the Einstein field equations. The approximation is based on a conformally flat truncation, which excludes gravitational radiation, as well as a removal of some additional gravitational field energy. After these two simplifications, only source terms proportional to Dirac delta distributions remain in the constraint equations. The skeleton Hamiltonian is exact in the test-body limit, it leads to the Einsteinian dynamics up to the first post-Newtonian approximation, and in the time-symmetric limit it gives the energy of the Brill-Lindquist solution exactly. The skeleton model for binary systems may be regarded as a kind of analytical counterpart to the numerical treatment of orbiting Misner-Lindquist binary black holes proposed by Gourgoulhon, Grandclément, and Bonazzola, even if they actually treat the corotating case. Along circular orbits, the two-black-hole skeleton solution is quasi-stationary and it fulfills the important property of equality of Komar and ADM masses. Explicit calculations for the determination of the last stable circular orbit of the binary system are performed up to the tenth post-Newtonian order within the skeleton model.

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I. INTRODUCTION

The description of the motion of binary black holes within the Einsteinian theory of gravity is quite a challenging problem and is likely to be finally solvable by numerical means only. However, it seems that the full success will occur in a far future. It has turned out in the past that analytical developments did always influence numerical investigations, hence further progress on the analytical side is certainly desirable. In this paper we shall present an analytical solution of truncated Einstein equations for systems of non-spinning point-like objects, which shows many aspects of past numerical models.

Most of the numerical computations of quasi-stationary initial data for black-hole binaries are based on the assumption that the space metric is conformally flat [1, 2, 3]. They often assume as well the existence of an approximate helical Killing vector which permits to elaborate some manageable formulations (also see [4]). Recently, the time-symmetric initial data of Brill-Lindquist [5] were generalized by adding a non-conformally flat contribution that incorporates pieces of information provided by perturbative post-Newtonian calculations [6]. The present paper does not follow this line, but rather keeps the spirit of former numerical simulations. We essentially perform two simplifications: on the one hand, we adopt the often-used assumption of conformally flat space metric; on the other hand, we attribute the non-linear gravitational field energy to point-like sources, which are modeled as Dirac delta distributions. We solve the resulting truncated Einstein equations analytically and get what we call, in reference to the point-like character of the source support, the *skeleton* solution. The calculations are carried out for $\mathcal{N}$ bodies in a mathematically sound way by working from the beginning in $(d+1)$-dimensional space-time and taking the limit $d \to 3$ in the final expressions, so that the Dirac delta distributions modeling the sources are consistently handled [7].

Our space metric and extrinsic curvature may be regarded as some generalization of the multiple black hole Brill-Lindquist initial value solution [5] to arbitrary black-hole momenta preserving its original asymptotic structure. Moreover, the energy function can be used as generator of the dynamical evolution of the system. We shall refer to it as the skeleton Hamiltonian associated to the full Arnowitt-Deser-Misner (ADM) Hamiltonian. It agrees with the exact Hamiltonian *(i)* in the test body limit, *(ii)* at the first post-Newtonian (1PN) approximation, i.e., at the first order in powers of $1/c^2$ (c denoting the speed of light) beyond the Newtonian dynamics, and *(iii)* in the limit of special relativity in absence of gravity. The skeleton gravitational field is given explicitly, as well as the equations of motion.

As a first application, we shall investigate the location of the last stable (circular) orbit (LSO) for binary systems. In order to reach a proper accuracy, the last-

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stable-orbit parameters are worked out up to the tenth post-Newtonian (10PN) approximation within the skeleton model. The comparison of second and third post-Newtonian results derived from the skeleton model with the corresponding quantities computed in the full Einstein theory delivers a quantitative measure for the missing terms in the skeleton approach.

II. DESCRIPTION OF THE BLACK-HOLE DYNAMICS IN ADM FORMALISM

We are interested in a system of $\mathcal{N}$ black holes interacting gravitationally. Its description will be achieved within the ADM Hamiltonian framework using a $(d+1)$ splitting of space-time. The spatial indices are denoted by Latin letters and vary from 1 to d, whereas the space-time indices are Greek and vary from 0 to d. In a given chart, $(\mathbf{x} = x^i,\, t = x^0/c)$, the motion of the Ath black hole is described by a curve $x^i_A(t)$, for $A = 1, \ldots, \mathcal{N}$. The conjugate momentum of the position x^i_A is called p_{Ai}. The constant mass parameter of the Ath black hole is denoted by m_A. The Hamiltonian variables associated to the gravitational field are taken to be (i) the space metric γ_{ij} induced by the full space-time metric $g_{\mu\nu}$ on the hypersurface $t = \text{const}$, and (ii) its conjugate momentum $(c^3/16\pi G)\pi^{ij}$ where G is the Newtonian gravitational constant. The other components of $g_{\mu\nu}$ are parametrized by the lapse and shift functions: $N \equiv (g_{0i}g_{0j}\gamma^{ij} - g_{00})^{\frac{1}{2}}$ and $N_i \equiv g_{0i}$, with γ^{ij} standing for the inverse of γ_{ij}. It is also useful to introduce the shift vector $N^i \equiv \gamma^{ij}N_j$. The spatial indices will be raised and lowered with the help of γ^{ij} and γ_{ij}, respectively.

We now assume that the "center-of-field" $\mathbf{x}_A(t)$ of object A can be modeled in the fictitious space-time associated with the flat metric by means of the Dirac distribution $\delta_A \equiv \delta(\mathbf{x} - \mathbf{x}_A)$. This prescription is known to lead to the Brill-Lindquist Hamiltonian when the black hole velocities are set to zero, provided the mass parameters m_A, $A = 1, \ldots, \mathcal{N}$, are identified with the Brill-Lindquist masses [8, 9]. It also leads to the correct 2PN dynamics [10, 11] for compact binaries, which can itself be recovered by means of the extended-body approach [12]. Moreover, the singularities generated by the distributions δ_A have shown to be curable by dimensional regularization at the 3PN order [7]. Similarly, we shall see in the present work that, for an appropriate choice of the space dimension d, the δ_A-type sources do not generate any ambiguity, so that the 3-dimensional quantities can be obtained by dimensional regularization. Finally, this point-particle approach allows to bypass delicate investigations about the behavior of the gravitational field near the objects.

In an asymptotically flat space-time of dimension $d+1$, the Hamiltonian of $\mathcal{N}$ point masses provided by the ADM formalism reads [13, 14, 15]

$$H = \frac{c^4}{16\pi G}\left(\int d^d\mathbf{x}\,\left(N\mathcal{H} + N^i\mathcal{J}_i\right) + \int d^d\mathbf{x}\;\partial_i(\partial_j\gamma_{ij} - \partial_i\gamma_{jj})\right), \quad (2.1)$$

where $\partial_i \equiv \partial/\partial x^i$. The super-Hamiltonian density $\mathcal{H}$ appearing under the integral sign is defined by

$$\mathcal{H} \equiv -\mathrm{R}\sqrt{\gamma} + \frac{1}{\sqrt{\gamma}}\left(\pi^i_{\ j}\pi^j_{\ i} - \frac{1}{d-1}(\pi^i_{\ i})^2\right) + \frac{16\pi G}{c^2}\sum_{A=1}^{\mathcal{N}}\left(m_A^2 + \frac{\gamma^{ij}p_{Ai}p_{Aj}}{c^2}\right)^{\frac{1}{2}}\delta_A, \quad (2.2a)$$

while the supermomentum density $\mathcal{J}_i$ is given by

$$\mathcal{J}_i \equiv -2\partial_j\pi^j_{\ i} + \pi^{kl}\partial_i\gamma_{kl} - \frac{16\pi G}{c^3}\sum_{A=1}^{\mathcal{N}}p_{Ai}\delta_A. \quad (2.2b)$$

In Eq. (2.2a) R denotes the space curvature of the hypersurface $t = \text{const}$ and γ is the determinant of the matrix γ_{ij}. As the lapse and shift functions are Lagrange multipliers, the variation of the Hamiltonian (2.1) with respect to them leads to the constraint equations

$$\mathcal{H} = 0, \quad \mathcal{J}_i = 0. \quad (2.3)$$

Before solving Eqs. (2.3), we must first fix the coordinate system. It is convenient to work in the so-called ADM transverse trace-free (ADMTT) gauge [16, 17] defined by the two conditions

$$\partial_j\left(\gamma_{ij} - \frac{1}{d}\delta_{ij}\gamma_{kk}\right) = 0, \quad \pi^{ii} = 0. \quad (2.4)$$

In this gauge, the metric decomposes into its trace γ_{kk}, which may be parametrized as $d\,\Psi^{4/(d-2)}$ (in order to get a simple expression for the curvature density), and a transverse trace-free part h^{TT}_{ij}. We thus have

$$\gamma_{ij} = \Psi^{\frac{4}{d-2}}\delta_{ij} + h^{\mathrm{TT}}_{ij}, \quad \partial_j h^{\mathrm{TT}}_{ij} = 0, \quad h^{\mathrm{TT}}_{ii} = 0, \quad (2.5a)$$

$$\pi^{ii} = 0. \quad (2.5b)$$

The momentum π^{ij} needs itself to be split into a transverse trace-free contribution π^{ij}_{TT}, and a rest $\tilde{\pi}^{ij}$,

$$\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ij}_{\mathrm{TT}}. \quad (2.6)$$

These two terms are uniquely defined assuming they decay as $1/(x^ix^i)^{\frac{d-1}{2}}$ at spatial infinity and are given as a combination of derivatives of the Poisson integral $\Delta^{-1}\partial_j\pi^{ij}$, and the Poisson integral of the latter quantity, namely $\Delta^{-2}\partial_j\pi^{ij}$.

We can then try to compute Ψ and $\tilde{\pi}^{ij}$ from the two constraint equations (2.3) supplemented by the boundary conditions at spatial infinity resulting from the asymptotic flatness of space-time; notably $\Psi - 1 \sim (x^kx^k)^{\frac{d-2}{2}}$,

$\tilde{\pi}^{ij}{\sim}1/(x^k x^k)^{\frac{d-1}{2}}$. It is actually more convenient to solve the Hamiltonian constraint for the quantity $\phi \equiv \frac{4(d-1)}{d-2}(\Psi - 1)$ rather than Ψ to get rid off the spurious constant at spatial infinity. We thus have

$$\Psi = 1 + \frac{d-2}{4(d-1)}\phi\,.$$

After solving the constraint equations (2.3), we insert the ensuing expressions for ϕ and $\tilde{\pi}^{ij}$ into the right-hand side of Eq. (2.1). We get a *reduced* Hamiltonian depending on x^i_A, p_{Ai}, $h^{\rm TT}_{ij}$, and $\pi^{ij}_{\rm TT}$ only:

$$\begin{aligned} H_{\rm red} &= H_{\rm red}[x^i_A, p_{Ai}, h^{\rm TT}_{ij}, \pi^{ij}_{\rm TT}] \\ &= \frac{c^4}{16\pi G}\int d^d\mathbf{x}\; \partial_i(\partial_j\gamma_{ij} - \partial_i\gamma_{jj}) \\ &= -\frac{c^4}{16\pi G}\int d^d\mathbf{x}\; \Delta\phi. \end{aligned} \tag{2.7}$$

This Hamiltonian contains the full information about the evolution of the matter (x^i_A, p_{Ai}) and field $(h^{\rm TT}_{ij}, \pi^{ij}_{\rm TT})$ variables [14, 17]. As the change of variables $(\gamma_{ij}, \pi^{ij}) \to (h^{\rm TT}_{ij}, \pi^{ij}_{\rm TT})$ is not canonical, the Poisson brackets of $h^{\rm TT}_{ij}$ and $\pi^{ij}_{\rm TT}$ take a form involving a transverse trace-free projection operator [14, 17].

III. SKELETON HAMILTONIAN

The reduced Hamiltonian $H_{\rm red}$ provides the equations of motion of the $\mathcal{N}$ bodies together with the evolution equations of the gravitational field. As they are strongly coupled it is hopeless to solve them analytically, but they can be solved perturbatively using, notably, the post-Newtonian iterative scheme [17, 18, 19]. The resulting dynamics is highly reliable as long as the typical speeds remain smaller than, say, about $0.3c$. When binary black holes reach their last stable orbits, the reliability of the approximation breaks down since the system enters a strong field regime. The only way to understand precisely this crucial stage is to perform numerical simulations in full general relativity. Our purpose here is to search for a simple non-perturbative model applicable to arbitrary strong fields, eventually leaving the Einsteinian theory but always staying close enough to it, in order to give an element of comparison with numerical investigations. The space metric obtained in our approach should also be usable as initial condition.

We shall adopt here a Wilson-Mathews-type prescription [20, 21], setting the non-conformally flat part of γ_{ij}, namely $h^{\rm TT}_{ij}$, equal to zero. As $h^{\rm TT}_{ij}$ is of order $1/c^4$, the difference between our γ_{ij} and the Einsteinian space metric shows up at the 2PN level. For spherically symmetric space-times the difference is zero. In the case where the present model is used to fix initial Cauchy data, the condition of a conformally flat space metric needs to hold on only one hypersurface, say $\Sigma_0 : t = t_0$. Choosing $h^{\rm TT}_{ij}$ to be zero on Σ_0 is always possible, though it is not physically realistic in general for isolated self-gravitating systems. Our attitude in this instance will be to accept the resulting error. In return, we shall be able to determine the field γ_{ij} without assuming zero velocities of the bodies at initial time.

We thus demand here that

$$h^{\rm TT}_{ij} = 0, \tag{3.1}$$

and the problem reduces to solve the constraints $\mathcal{H} = 0$, $\mathcal{J}_i = 0$ under this condition. The space metric γ_{ij} is then proportional to δ_{ij}, which implies that in the supermomentum $\mathcal{J}_i$ the term $\pi^{kl}\partial_i\gamma_{kl}$ vanishes [cf. Eq. (2.2b)]. The field momentum π^{ij} relates to π^i_j through the identities

$$\pi^{ij} = \gamma^{jk}\pi^i_k = \Psi^{-\frac{4}{d-2}}\pi^i_j\,. \tag{3.2}$$

In particular, π^i_j is symmetric and trace-free (STF) as is π^{ij}. The determinant γ appearing in $\mathcal{H}$ is merely

$$\gamma = \Psi^{\frac{4d}{d-2}}\,. \tag{3.3}$$

Finally, the 3-curvature density $\mathrm{R}\sqrt{\gamma}$ takes a remarkably simple form [22]:

$$\mathrm{R}\sqrt{\gamma} = -\Psi\Delta\phi\,. \tag{3.4}$$

Collecting these relations together with the constraint equations (2.3), we find:

$$\begin{aligned} \Delta\phi = &-\Psi^{-\frac{3d-2}{d-2}}\pi^i_j\pi^j_i \\ &-\frac{16\pi G}{c^2}\sum_{A=1}^{\mathcal{N}} m_A\delta_A\Psi^{-1}\left(1 + \Psi^{-\frac{4}{d-2}}\frac{p_A^2}{m_A^2c^2}\right)^{\frac{1}{2}}, \end{aligned} \tag{3.5a}$$

$$\partial_j\pi^j_i = -\frac{8\pi G}{c^3}\sum_{A=1}^{\mathcal{N}} p_{Ai}\delta_A\,, \tag{3.5b}$$

with $p_A^2 \equiv p_{Ai}p_{Ai}$.

In the system (3.5) the second equation decouples from the first one and is solved by searching for a particular solution proportional to the symmetric trace-free part of the derivative of a certain vector potential V_i; namely,

$$\pi^{i\;(\rm part)}_j = \mathrm{STF}(2\partial_iV_j) \equiv \partial_iV_j + \partial_jV_i - \frac{2}{d}\delta_{ij}\partial_kV_k\,. \tag{3.6}$$

The equation for the divergence ∂_kV_k is obtained by applying the operator $\partial_{ij} \equiv \partial_i\partial_j$ simultaneously on the two sides of (3.6), hence $2\Delta\partial_kV_k = d/(d-1)\partial_{ij}\pi^i_j$. This yields immediately $\partial_j\pi^j_i = \Delta V_i + (\frac{d}{2} - 1)/(d-1)\Delta^{-1}\partial_{ijk}\pi^j_k$, and from the constraint (3.5b) it follows that:

$$\Delta V_i = -\frac{8\pi G}{c^3}\sum_{A=1}^{\mathcal{N}}\left(p_{Ai}\delta_A - \frac{d-2}{2(d-1)}p_{Aj}\partial_{ij}\Delta^{-1}\delta_A\right). \tag{3.7}$$

We now notice that the d-dimensional Laplacian of the mth power of $r_A \equiv [(x^i - x^i_A)(x^i - x^i_A)]^{\frac{1}{2}}$ satisfies

$$\Delta r_A^m = m(m-2+d) r_A^{m-2} \tag{3.8}$$

in the sense of functions, and that, in the sense of distributions,

$$\Delta r_A^{2-d} = -\frac{4\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d-2}{2}\right)} \delta_A \,. \tag{3.9}$$

As a result we may choose:

$$V_i = \frac{G}{c^3} \frac{\Gamma(\frac{d-2}{2})}{2\pi^{\frac{d-2}{2}}} \sum_{A=1}^{\mathcal{N}} \left(\frac{4 p_{Ai}}{r_A^{d-2}} - \frac{(d-2) p_{Aj}}{(d-1)(4-d)} \partial_{ij} r_A^{4-d} \right), \tag{3.10}$$

which is nothing but the 1PN vector potential $V^i_{(3)}$ of Ref. [7] verifying $\pi^{ij} = \mathrm{STF}(2\partial_i V^j_{(3)}) + \mathcal{O}(1/c^5)$.

Our second simplification consists in stating that the homogeneous solution of Eq. (3.5b), $\pi^{i\,(\mathrm{hom})}_{\ j} \equiv \pi^i_{\ j} - \pi^{i\,(\mathrm{part})}_{\ j}$, is precisely zero, which implies $\pi^{i\ \mathrm{TT}}_{\ j} = 0$. When this prescription is imposed on one unique hypersurface, it merely represents a particular choice of initial data. The resulting field momentum $\pi^i_{\ j} = \mathrm{STF}(2\partial_i V_j)$ is identical to the rescaled extrinsic curvature proposed by Bowen and York in their time-asymmetric formulation of the Cauchy problem [23] and adopted e.g. in the puncture method [24] or various calculations of the last stable circular orbit [25, 26, 27]. Although it is a mathematical solution of the constraint equations for a conformally flat metric, it is not realistic in the sense that it does not result from an evolution process in absence of incoming radiation. It does not coincide either with the Wilson-Mathews momentum $(\pi^i_{\ j})_{\mathrm{WM}}$ derived from the condition $\dot{h}^{\mathrm{TT}}_{ij} = 0$:

$$(\pi^i_{\ j})_{\mathrm{WM}} = (1+\phi)\pi^{i\,(\mathrm{part})}_{\ j} + (d-1)\tilde{\pi}^{ij}_{(5)} + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{3.11}$$

where the quantity $\tilde{\pi}^{ij}_{(5)}$, defined as the $1/c^5$ part of the solution [matching the form (3.6)] of the equation $(d-1)\partial_j \tilde{\pi}^{ij}_{(5)} = -\partial_j(\phi \pi^{j\,(\mathrm{part})}_{\ i})$, is explicitly given in paper [18] for $d = 3$. The difference $(\pi^i_{\ j})_{\mathrm{WM}} - \pi^{i\ (\mathrm{part})}_{\ j}$ is only of order $1/c^5$, which corresponds to the 2PN approximation. On the Hamiltonian level, it is of 3PN order only. One might then argue that the Wilson-Mathews condition $h^{\mathrm{TT}}_{ij} = 0$, valid for all times, i.e. particularly $\dot{h}^{\mathrm{TT}}_{ij} = 0$, is in contradiction to the condition $\pi^{i\ \mathrm{TT}}_{\ j} = 0$. However, it is not the case because the momentum constraint equation is linear in $\pi^i_{\ j}$ and decouples from any other field equation when the space metric is conformally flat, so that the quantity obtained by adding an arbitrary homogeneous solution to $\pi^i_{\ j}$ still satisfies the constraint. As our $\pi^i_{\ j}$ differs from $(\pi^i_{\ j})_{\mathrm{WM}}$ by such a homogeneous solution (i.e. a transverse-trace free object), there is no contradiction between the two approaches.

The integration of the Hamiltonian constraint (3.5a) is much more involved, principally because of the appearance of ϕ in the source and the presence of a non-compact support term. On the contrary, the compact support term is harmless. It is made of a sum of Dirac distributions δ_A that formally determine the boundary conditions near the black holes. By subtracting from the conformal factor the $1/r_A^{d-2}$ terms entering ϕ, we get an object that satisfies an equation whose source is entirely free of δ_A's (this is the essence of paper [24]). By contrast, we shall keep here the constraint equation (3.5a) under its original form in order to determine the dependence of the coefficient of $1/r_A^{d-2}$ on the positions x^i_A and momenta p_{Ai}.

To solve the Hamiltonian constraint (3.5a) iteratively, we have to face the delicate computation of $\Delta^{-1}\left(\Psi^{\frac{2-3d}{d-2}} \pi^i_{\ j} \pi^j_{\ i}\right)$. The expansion of the prefactor $\Psi^{\frac{2-3d}{d-2}} = \left(1 + \frac{d-2}{4(d-1)}\phi\right)^{\frac{2-3d}{d-2}}$ in powers of ϕ is known to give rise to poles when $d \to 3$ at the level corresponding to the 3PN approximation [7]. In full general relativity, these poles cancel in the Hamiltonian with similar quantities coming from h^{TT}_{ij}, but such a cancellation cannot occur in our case. In particular, *the 3PN approximation of the Wilson-Mathews model does not exist* for binary black-holes. We must therefore treat the π^2 term differently. Let us first note that this term enters the space metric at the 3PN order only, since $\pi^i_{\ j}\pi^j_{\ i} = \mathcal{O}(1/c^6)$, but contributes to the 1PN Hamiltonian [because of the global factor $c^4/(16\pi G)$ in Eq. (2.7)]. To be consistent with the accuracy of the former assumption $h^{\mathrm{TT}}_{ij} = 0$ which amounts to neglect some 2PN corrections in the Hamiltonian, we have to keep at least the leading term in the post-Newtonian expansion of $\left(1 + \frac{d-2}{4(d-1)}\phi\right)^{\frac{2-3d}{d-2}} \pi^i_{\ j}\pi^j_{\ i}$, modulo a possible total space derivative. Now, the tensor density $\pi^i_{\ j}$ is itself proportional to the symmetric trace-free part of the space derivative $\partial_i V_j$, so that

$$\pi^i_{\ j}\pi^j_{\ i} = 2\mathrm{STF}(\pi^i_{\ j})\mathrm{STF}(\partial_i V_j) = 2\pi^i_{\ j}\partial_i V_j \,.$$

As a consequence, the latter expression can be split into a "skeleton" term $-2V_j \partial_i \pi^i_{\ j}$ with compact support, and a "flesh" term $\partial_i(2V_j \pi^i_{\ j})$ which takes into account the field between the point particles. The flesh term contributes to the Hamiltonian through the quantity:

$$\frac{c^4}{16\pi G} \int d^d\mathbf{x}\, \Psi^{\frac{2-3d}{d-2}} \partial_i(2V_j \pi^i_{\ j}) = \frac{c^4}{32\pi G} \frac{3d-2}{d-1} \int d^d\mathbf{x} \left(\Psi^{\frac{4(1-d)}{d-2}} V_j \pi^i_{\ j} \partial_i \phi \right).$$

We are allowed to regard it as negligible in our scheme and keep the skeleton term alone in the decomposition of $\pi^i_{\ j}\pi^j_{\ i}$. Our third simplification consits thus in performing the substitution

$$\Psi^{\frac{2-3d}{d-2}}\pi^i_j\pi^j_i \to -2\Psi^{\frac{2-3d}{d-2}}V_j\partial_i\pi^i_j$$

$$= \frac{16\pi G}{c^3}\Psi^{\frac{2-3d}{d-2}}\sum_{A=1}^{\mathcal{N}} p_{Aj}V_j\delta_A\,. \quad (3.12)$$

After the replacement (3.12) is made, the right-hand side of Eq. (3.5a) takes the form $\sum_A f_A(\mathbf{x})\delta_A$, where the f_A's are some unknown functions. For d belonging to an appropriate range of values, f_A is regular at $\mathbf{x}=\mathbf{x}_A$, and $\Delta\phi = \sum_A f_A(\mathbf{x}_A)\delta_A$. We see that in our model the source of the Hamiltonian constraint reduces to some "skeleton" made of a linear combination of Dirac distributions. For this reason, all quantities computed in the present approximation will be referred to as skeleton quantities. Defining the $\mathcal{N}$ quantities α_A by the relation

$$-\frac{16\pi G}{c^2}\alpha_A \equiv f_A(\mathbf{x}_A), \quad A=1,\ldots,\mathcal{N},$$

the function ϕ can be written as

$$\phi = \frac{4G}{c^2}\frac{\Gamma(\frac{d-2}{2})}{\pi^{\frac{d-2}{2}}}\sum_{A=1}^{\mathcal{N}}\frac{\alpha_A}{r_A^{d-2}}\,, \quad (3.13)$$

and the conformal factor reads

$$\Psi = 1 + A_d\sum_{A=1}^{\mathcal{N}}\frac{\alpha_A}{r_A^{d-2}}\,, \quad (3.14)$$

where we have posed, in order to lighten the notation,

$$A_d \equiv \frac{G}{c^2}\frac{d-2}{d-1}\frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}\,.$$

It is also useful to introduce the (regularized) value of Ψ at the particle position $\mathbf{x}_A$:

$$\Psi_A \equiv \Psi(\mathbf{x}=\mathbf{x}_A) = 1 + A_d\sum_{B\neq A}\frac{\alpha_B}{r_{AB}^{d-2}},$$

where $r_{AB} \equiv [(x^i_A - x^i_B)(x^i_A - x^i_B)]^{\frac{1}{2}}$.

After making the replacement (3.12) on the right-hand side of Eq. (3.5a), inserting there expressions (3.13) and (3.14), we arrive at

$$-\frac{c^2}{16\pi G}\Delta\phi = \sum_{A=1}^{\mathcal{N}}\alpha_A\delta_A = \sum_{A=1}^{\mathcal{N}}\left\{m_A\left(1+A_d\sum_{B\neq A}\frac{\alpha_B}{r_{AB}^{d-2}}\right)^{-1}\left[1+\left(1+A_d\sum_{C\neq A}\frac{\alpha_C}{r_{AC}^{d-2}}\right)^{-\frac{4}{d-2}}\frac{p_A^2}{m_A^2c^2}\right]^{\frac{1}{2}} + \left(1+A_d\sum_{B\neq A}\frac{\alpha_B}{r_{AB}^{d-2}}\right)^{\frac{2-3d}{d-2}}\frac{p_{Ai}V_{Ai}}{c}\right\}\delta_A, \quad (3.15)$$

with $V_{Ai} \equiv V_i(\mathbf{x}=\mathbf{x}_A)$. Making use of Eq. (3.10), we see that

$$p_{Ai}V_{Ai} = \frac{A_d}{2c}\sum_{B\neq A}\left(\frac{3d-2}{d-2}(p_{Ai}\,p_{Bi}) + (d-2)(n^i_{AB}\,p_{Ai})(n^j_{AB}\,p_{Bj})\right)r_{AB}^{2-d}. \quad (3.16)$$

Because of the linear independence of the distributions δ_A, the Hamiltonian constraint (3.15) is finally equivalent to a system of $\mathcal{N}$ algebraic equations which reads

$$\alpha_A = \frac{m_A}{1+A_d\sum_{B\neq A}\frac{\alpha_B}{r_{AB}^{d-2}}}\left[1+\frac{p_A^2/(m_A^2c^2)}{\left(1+A_d\sum_{C\neq A}\frac{\alpha_C}{r_{AC}^{d-2}}\right)^{\frac{4}{d-2}}}\right]^{\frac{1}{2}} + \frac{p_{Ai}V_{Ai}/c}{\left(1+A_d\sum_{B\neq A}\frac{\alpha_B}{r_{AB}^{d-2}}\right)^{\frac{3d-2}{d-2}}}, \quad A=1,\ldots,\mathcal{N}. \quad (3.17)$$

Solving the above system provides in principle the values of all the α_A's. These quantities represent some effective masses, as suggested by the approximate equality $\alpha_A = m_A + \mathcal{O}(1/c^2)$. They are functions of $x^i_A - x^i_1$ (for $A\neq 1$) and p_{Ai} [defined implicitly by Eqs. (3.17)] .

The skeleton Hamiltonian reads

$$H = -\frac{c^4}{16\pi G}\int d^d\mathbf{x}\;\Delta\phi = c^2\sum_{A=1}^{\mathcal{N}}\alpha_A\,, \quad (3.18)$$

which shows that the ADM mass $M_{\mathrm{ADM}} = H/c^2$ is just the sum of the $\mathcal{N}$ effective masses α_A. The Hamiltonian (3.18) depends only on the matter variables: $H = H(x^i_A - x^i_1, p_{Ai})$, so that the system is automatically conservative. The space metric is deduced from Eqs. (2.5a), (3.1), and (3.14):

$$\gamma_{ij} = \left(1+A_d\sum_{A=1}^{\mathcal{N}}\frac{\alpha_A}{r_A^{d-2}}\right)^{\frac{4}{d-2}}\delta_{ij}. \quad (3.19)$$

By construction, H and γ_{ij} coincide with their counterparts in the Einsteinian theory up to the 1PN order. The test-body limit is achieved by taking the mass of one black hole, say of label 1, much larger than the masses of its companion: $m_1 \gg m_A$, where $A = 2, \ldots, \mathcal{N}$. To get the test-body Hamiltonian exactly, one has to let formally m_1 go to infinity ($m_1 \to +\infty$) in the Hamiltonian (3.18) while keeping m_1/r_{1A}^{d-2}, $A = 2, \ldots, \mathcal{N}$, constant as well as all linear momenta p_{Ai}. It follows that particularly the flesh term $\partial_i(2V_j\pi^i_{\ j})$ does not contribute to this approximation. In absence of gravity ($G \to 0$) our Hamitlonian has the form of the exact special-relativistic Hamitlonian for the system of $\mathcal{N}$ free point particles. When $p_{Ai} = 0$ for all A, we recover the Brill-Lindquist solution, for which Eqs. (3.17) reduce to $\alpha_A(1+A_d\sum_{B\neq A}\alpha_B/r_{AB}^{d-2}) = m_A$. As a consequence our model also generalizes the Brill-Lindquist data.

Let us observe that the particle masses m_A cannot be equated to the ADM masses computed at the spatial infinities of the $\mathcal{N}$ black hole sheets when one or several momenta are non-zero. When such an identification is performed, it leads automatically to the same relation between α_A and m_A as in the zero momentum case, whereas the correct dynamics up to the 1PN order is known to be the one that derives from the Hamiltonian (3.18) determined by the set of Eqs. (3.17) with $p_{Ai} \neq 0$. The actual dependence of the effective masses on the m_A's must take the kinetic energy of the system into account. This indicates a relative boost of the black hole sheets with respect to each others. The issue is complicated by the fact that the Bowen-York data for one single black hole do not result from a boosted Schwarzschild field [28]. This point, being still rather unclear, needs to be investigated in detail. We do not intend to discuss it here.

IV. SKELETON GRAVITATIONAL FIELD

In the ADM formalism, the evolution of the space metric in $(d+1)$ dimensions is given by [cf. Eq. (4.70) in Ref. [15]][1]

$$\frac{1}{c}\partial_t\gamma_{ij} = \frac{N}{\sqrt{\gamma}}(2\pi_{ij} - \pi^k_{\ k}\gamma_{ij}) + 2D_{(i}N_{j)}\,, \tag{4.1}$$

where the operator D_i denotes the space covariant derivative. Its explicit action on the shift function N_i after symmetrization is

$$D_{(i}N_{j)} = \gamma_{k(i}\partial_{j)}N^k + \frac{1}{2}N^k\partial_k\gamma_{ij}\,. \tag{4.2}$$

For the conformally flat 3-metric $\gamma_{ij} = \Psi^{\frac{4}{d-2}}\delta_{ij}$, we have $\gamma_{ij} - \frac{1}{d}\gamma_{kk}\delta_{ij} = 0$, hence

$$\partial_t\left(\gamma_{ij} - \frac{1}{d}\gamma_{kk}\delta_{ij}\right) = 0\,.$$

After replacing in the above equation $\partial_t\gamma_{ij}$ and $\partial_t\gamma_{kk}$ by the expressions following from Eq. (4.1) and dropping all contributions proportional to $\pi^k_{\ k} = 0$, we arrive at:

$$\mathrm{STF}(\partial_i N^j) = -\Psi^{-\frac{2d}{d-2}}N\pi^i_{\ j}\,. \tag{4.3}$$

The evolution equation for the field momenta π^{ij} is known in d dimensions as well [cf. Eq. (4.71) in Ref. [15]]. It reads

$$\frac{1}{c}\partial_t\pi^{ij} = -\sqrt{\gamma}\left[N\left(\mathrm{R}^{ij} - \frac{1}{2}\gamma^{ij}\mathrm{R}\right) - D^iD^jN + \gamma^{ij}D_mD^mN\right] + \frac{N}{\sqrt{\gamma}}\left[\pi^{ij}\pi^k_{\ k} - 2\pi^i_{\ k}\pi^{kj} + \frac{1}{2}\gamma^{ij}\left(\pi^{kl}\pi_{kl} - \frac{1}{2}(\pi^k_{\ k})^2\right)\right]$$
$$- \left[\pi^{kj}D_kN^i + \pi^{ki}D_kN^j - D_k(\pi^{ij}N^k)\right] + \frac{8\pi G}{c^4}N\sum_{A=1}^{\mathcal{N}}\frac{p_{Ak}p_{Al}}{m_A}\gamma^{ik}\gamma^{jl}\left(1 + \frac{p_{Am}p_{An}}{m_A^2c^2}\gamma^{mn}\right)^{-\frac{1}{2}}\delta_A\,. \tag{4.4}$$

In the ADM gauge, the trace (with respect to flat metric δ_{ij}) of (4.4) reduces to

$$\Psi^2\left(\frac{d-2}{2}N\mathrm{R} - (d-1)\Psi^{-\frac{4}{d-2}}D_iD_iN\right) - \frac{4-d}{2}\Psi^{-\frac{2(d+2)}{d-2}}N\pi^i_{\ j}\pi^j_{\ i} - 2\pi^{kl}D_kN^l$$
$$+ \frac{8\pi G}{c^4}N\Psi^{-\frac{8}{d-2}}\sum_{A=1}^{\mathcal{N}}\frac{p_A^2}{m_A}\left(1 + \frac{p_A^2}{m_A^2c^2}\Psi^{-\frac{4}{d-2}}\right)^{-\frac{1}{2}}\delta_A = 0\,. \tag{4.5}$$

Making now use of Eqs. (3.4), (3.5a), (4.3), and of the two useful relations $D_{(i}D_{j)}N = \gamma_{k(i}\partial_{j)}(D^kN) + \frac{1}{2}\partial_k\gamma_{ij}D^kN$ and $D_iD_iN = 2\partial_iN\partial_i\ln\Psi + \Delta N$, Eq. (4.5) can be rewritten as

[1] We use here and below the usual notation for symmetrization: $2D_{(i}N_{j)} \equiv D_iN_j + D_jN_i$.

$$- (d-1)\Delta(N\Psi) + \frac{3d-2}{4}\Psi^{-\frac{3d-2}{d-2}} N\pi^i_{\ j}\pi^j_{\ i} + \frac{4\pi G}{c^2}(d-2)\Psi^{-1}N\sum_{A=1}^{\mathcal{N}} m_A\left(1+\frac{p_A^2}{m_A^2c^2}\Psi^{-\frac{4}{d-2}}\right)^{\frac{1}{2}}\delta_A$$
$$+ \frac{8\pi G}{c^4}\Psi^{-\frac{d+2}{d-2}}N\sum_{A=1}^{\mathcal{N}}\frac{p_A^2}{m_A}\left(1+\frac{p_A^2}{m_A^2c^2}\Psi^{-\frac{4}{d-2}}\right)^{-\frac{1}{2}}\delta_A = 0\,. \quad (4.6)$$

The form of the latter equation suggests to solve it for the auxiliary function $\chi \equiv N\Psi$ rather than for the lapse N [9]. At this stage, we apply to the right-hand side of Eq. (4.6) the replacement (3.12) which amounts to deleting a flesh term in the product $\Psi^{-\frac{3d-2}{d-2}}\pi^i_{\ j}\pi^j_{\ i}$. Since the Newtonian potential comes from the $1/c^2$ part of the lapse, the term $\Psi^{-1}\Psi^{-\frac{3d-2}{d-2}}\pi^i_{\ j}\pi^j_{\ i} = \mathcal{O}(1/c^6)$ is indeed a 2PN quantity and may be neglected consistently with our approximation. Finally, we obtain a Poisson equation for χ with the source term being some linear combination of the Dirac deltas:

$$\Delta\chi = \frac{4\pi G}{c^2}\frac{d-2}{d-1}\sum_{A=1}^{\mathcal{N}}\chi_A\Bigg(\frac{3d-2}{d-2}\frac{p_{Ai}V_{Ai}}{c}\Psi_A^{-\frac{4(d-1)}{d-2}}$$
$$+ m_A\Psi_A^{-2}\left(1+\frac{p_A^2}{m_A^2c^2}\Psi_A^{-\frac{4}{d-2}}\right)^{-\frac{1}{2}}\left(1+\frac{d}{d-2}\frac{p_A^2}{m_A^2c^2}\Psi_A^{-\frac{4}{d-2}}\right)\Bigg)\delta_A\,. \quad (4.7)$$

Following the same argument as in the derivation of the conformal factor in Sec. III, we may write

$$\chi = 1 - A_d\sum_{A=1}^{\mathcal{N}}\frac{\beta_A}{r_A^{d-2}}\,, \quad (4.8)$$

so that $\Delta\chi = \frac{4\pi G}{c^2}\frac{(d-2)}{(d-1)}\sum_{A=1}^{\mathcal{N}}\beta_A\delta_A$. We next replace the left-hand side of Eq. (4.7) by virtue of the latter relation, and equate the coefficients of δ_A. We are led to

$$\beta_A = \left(1 - A_d\sum_{B\neq A}\frac{\beta_B}{r_{AB}^{d-2}}\right)\Bigg(\frac{3d-2}{d-2}\frac{p_{Ai}V_{Ai}}{c}\Psi_A^{-\frac{4(d-1)}{d-2}}$$
$$+ m_A\Psi_A^{-2}\left(1+\frac{p_A^2}{m_A^2c^2}\Psi_A^{-\frac{4}{d-2}}\right)^{-\frac{1}{2}}$$
$$\times\left(1+\frac{d}{d-2}\frac{p_A^2}{m_A^2c^2}\Psi_A^{-\frac{4}{d-2}}\right)\Bigg), \quad A = 1,\ldots,\mathcal{N}. \quad (4.9)$$

This system of $\mathcal{N}$ equations for $\mathcal{N}$ unknowns provides in principle the values of the monopoles β_A entering the expression of χ, Eq. (4.8). The lapse function itself is then given by

$$N = \frac{\chi}{\Psi}\,.$$

For $\mathcal{N} = 2$ and $d = 3$, we recover the result of Ref. [9] in the time-symmetric limit $p_{Ai} = 0$ (with $A = 1, 2$).

In order to solve Eq. (4.3) for the shift function N^i, we take the first and second order space derivatives of both sides. We obtain:

$$\Delta N^i + \frac{d-2}{d}\partial_{ij}N^j = -2\partial_j\left(\Psi^{-\frac{2d}{d-2}}N\pi^i_{\ j}\right)\,, \quad (4.10a)$$

$$\frac{d-1}{d}\Delta(\partial_i N^i) = -\partial_{ij}\left(\Psi^{-\frac{2d}{d-2}}N\pi^i_{\ j}\right)\,. \quad (4.10b)$$

The source term of the second equation is the sum of a rational function of r_A, $A = 1,\ldots,\mathcal{N}$, namely $-\partial_i\big(\partial_j(\Psi^{-\frac{2d}{d-2}}N)\pi^i_{\ j}\big)$, plus a distribution, $-\partial_i\big(\Psi^{-\frac{2d}{d-2}}N\partial_j\pi^i_{\ j}\big)$. The Poisson integral of the term with non-compact support is not explicitly known, but that of the distributional term is easy to compute. Now, the new "flesh" term does not contribute at the Newtonian level. It is therefore at least post-Newtonian and will eventually play a role in the dynamics only at the 2PN order; it will be consistently neglected in the present model. Thus, the substitution (3.12) will be supplemented by the replacement:

$$\partial_j\left(\Psi^{-\frac{2d}{d-2}}N\pi^j_{\ i}\right) \to \Psi^{-\frac{2d}{d-2}}N\partial_j\pi^j_{\ i}$$
$$= -\frac{8\pi G}{c^3}\Psi^{-\frac{2d}{d-2}}N\sum_{A=1}^{\mathcal{N}}p_{Ai}\delta_A\,. \quad (4.11)$$

Taking Eq. (4.11) into account, Eq. (4.10b) becomes

$$\Delta(\partial_i N^i) = \frac{8\pi G}{c^3}\frac{d}{d-1}\partial_i\left(\sum_{A=1}^{\mathcal{N}}\chi_A\Psi_A^{-\frac{3d-2}{d-2}}p_{Ai}\delta_A\right)\,. \quad (4.12)$$

The solution of (4.12) can be obtained with the help of formula (3.9). It reads

$$\partial_i N^i = -\frac{2G}{c^3}\frac{d}{d-1}\frac{\Gamma(\frac{d-2}{2})}{\pi^{\frac{d-2}{2}}}$$
$$\times\sum_{A=1}^{\mathcal{N}}\chi_A\Psi_A^{-\frac{3d-2}{d-2}}p_{Ai}\partial_i\left(\frac{1}{r_A^{d-2}}\right)\,. \quad (4.13)$$

The Poisson equation (4.10a) after inserting the above expression for $\partial_j N^j$ and performing the substitution (4.11) takes the form

$$\Delta N^i = \frac{2G}{c^3}\frac{d-2}{d-1}\frac{\Gamma(\frac{d-2}{2})}{\pi^{\frac{d-2}{2}}} \times \sum_{A=1}^{\mathcal{N}} \chi_A \Psi_A^{-\frac{3d-2}{d-2}} p_{Aj}\partial_{ij}\left(\frac{1}{r_A^{d-2}}\right) + \frac{16\pi G}{c^3}\sum_{A=1}^{\mathcal{N}} \chi_A \Psi_A^{-\frac{3d-2}{d-2}} p_{Ai}\delta_A . \tag{4.14}$$

By virtue of the formulas (3.8) and (3.9), the solution of Eq. (4.14) is simply

$$N^i = \frac{G}{c^3}\frac{\Gamma(\frac{d-2}{2})}{\pi^{\frac{d-2}{2}}}\sum_{A=1}^{\mathcal{N}} \chi_A \Psi_A^{-\frac{3d-2}{d-2}} \times \left(\frac{d-2}{(4-d)(d-1)} p_{Aj}\partial_{ij} r_A^{4-d} - 4\frac{p_{Ai}}{r_A^{d-2}}\right). \tag{4.15}$$

V. BLACK-HOLE BINARY EQUATIONS OF MOTION

For a black-hole binary, the number of coupled equations composing the system (3.17) that gives the effective masses reduces to two, and the sum $\sum_{C\neq A}\alpha_C/r_{AC}^{d-2}$ contains only one term. It is now convenient to regard the two quantities Ψ_A as the unknowns:

$$\Psi_A = 1 + A_d \sum_{C\neq A}\frac{\alpha_C}{r_{AC}^{d-2}} = 1 + A_d\frac{\alpha_B}{r_{12}^{d-2}}, \qquad A = 1,2, \quad B\neq A. \tag{5.1}$$

Let us restrict ourselves to the space dimension $d = 3$. The constant A_d reads then $A_3 = G/(2c^2)$, and the system (3.17) consists of two equations for the two unknowns Ψ_B:

$$\Psi_B = 1 + \frac{G m_A}{2r_{12}c^2\Psi_A}\left(1+\frac{p_A^2}{m_A^2c^2\Psi_A^4}\right)^{\frac12} + \frac{G p_{Ai}V_{Ai}}{2r_{12}c^3\Psi_A^7}, \qquad B = 1,2, \quad A\neq B . \tag{5.2}$$

It is clearly possible to decouple the two equations (5.2). In Appendix A we show that each Ψ_B is a root of a polynomial of order 200.

The equations of motion of a black-hole binary read

$$\dot{x}_A^i = \frac{\partial H}{\partial p_{Ai}}, \tag{5.3a}$$

$$\dot{p}_{Ai} = -\frac{\partial H}{\partial x_A^i}, \tag{5.3b}$$

where the skeleton Hamiltonian H expressed in terms of the quantities Ψ_A equals

$$H = \frac{2c^4 r_{12}}{G}(\Psi_1+\Psi_2-2). \tag{5.4}$$

Using the theorem of implicit differentiation, we put Eqs. (5.3) in a form apparently more convenient for numerical integration. We start by differentiating Eq. (5.2) with respect to the positions. We find (here $B = 1,2$ and $A\neq B$):

$$\partial_{1i}\Psi_B \equiv \frac{\partial\Psi_B}{\partial x_1^i} = \tilde{\eta}_{A1}^i - \zeta_A\partial_{1i}\Psi_A, \tag{5.5a}$$

$$\partial_{2i}\Psi_B \equiv \frac{\partial\Psi_B}{\partial x_2^i} = \tilde{\eta}_{A2}^i - \zeta_A\partial_{2i}\Psi_A, \tag{5.5b}$$

where

$$\tilde{\eta}_{A1}^i \equiv -\frac{G m_A n_{12}^i}{2c^2 r_{12}^2\Psi_A}\left(1+\frac{p_A^2}{m_A^2c^2\Psi_A^4}\right)^{\frac12} + \frac{G\partial_{1i}(p_{Aj}V_{Aj}/r_{12})}{2c^3\Psi_A^7}, \tag{5.6a}$$

$$\tilde{\eta}_{A2}^i \equiv -\tilde{\eta}_{A1}^i, \tag{5.6b}$$

$$\zeta_A \equiv -\frac{\partial\Psi_B(x_C^i, p_{Ci}, \Psi_C)}{\partial\Psi_A} = \frac{G m_A}{2c^2 r_{12}\Psi_A^2}\frac{1+3p_A^2/(m_A^2c^2\Psi_A^4)}{[1+p_A^2/(m_A^2c^2\Psi_A^4)]^{\frac12}} + \frac{7G\, p_{Aj}V_{Aj}}{2c^3 r_{12}\Psi_A^8} = \frac{\chi_B-1}{\chi_A}. \tag{5.6c}$$

Equality (5.6b) follows from the fact that the right-hand side of relation (5.2) depends on x_1^i and x_2^i only through the black-hole separation $r_{12}\equiv[(x_1^i-x_2^i)(x_1^i-x_2^i)]^{\frac12}$, and that we have $\partial_{1i}r_{12} = -\partial_{2i}r_{12}$. Solving Eqs. (5.5a) with respect to $\partial_{1i}\Psi_B$ and Eqs. (5.5b) with respect to $\partial_{2i}\Psi_B$, we find

$$\partial_{1i}\Psi_B = \frac{\tilde{\eta}_{A1}^i - \zeta_A\tilde{\eta}_{B1}^i}{1-\zeta_1\zeta_2}, \tag{5.7a}$$

$$\partial_{2i}\Psi_B = -\partial_{1i}\Psi_B . \tag{5.7b}$$

It remains to evaluate the momentum derivatives $\partial_{p_{1i}}\Psi_B \equiv \partial\Psi_B/\partial p_{1i}$ and $\partial_{p_{2i}}\Psi_B \equiv \partial\Psi_B/\partial p_{2i}$. The system of equations for $\partial_{p_{1i}}\Psi_B$ is given by

$$\partial_{p_{1i}}\Psi_B = \tilde{\theta}_{A1}^i - \zeta_A\partial_{p_{1i}}\Psi_A, \quad B = 1,2, \quad A\neq B, \tag{5.8}$$

where we have posed:

$$\tilde{\theta}_{A1}^i \equiv \frac{G}{2m_A c^4 r_{12}\Psi_A^5}\frac{p_{1i}\delta_{A1}}{\left(1+\frac{p_A^2}{m_A^2c^2\Psi_A^4}\right)^{\frac12}}$$

$$+\frac{G\,\partial_{p_{1i}}(p_{Aj}V_{Aj})}{2c^3r_{12}\Psi_A^7}. \tag{5.9}$$

The coefficients ζ_A are identical to those appearing in $\partial_{1i}\Psi_B$ because they represent the result of the differentiation of the right-hand side of relation (5.2) with respect to Ψ_A at x_A^i and p_{Ai} constant. The solution of Eqs. (5.8) with respect to $\partial_{p_{1i}}\Psi_B$ reads

$$\partial_{p_{1i}}\Psi_B = \frac{\tilde{\theta}^i_{A1} - \zeta_A\tilde{\theta}^i_{B1}}{1-\zeta_1\zeta_2}, \tag{5.10}$$

from which we deduce $\partial_{p_{2i}}\Psi_B$ by exchanging the particle labels.

It is convenient to introduce the rescaled variables

$$\eta^i_A \equiv \frac{2c^2r_{12}}{G}\,\tilde{\eta}^i_{AA}, \quad \theta^i_{AB} \equiv \frac{2c^4r_{12}}{G}\,\tilde{\theta}^i_{AB}, \quad A=1,2. \tag{5.11}$$

Making use of Eqs. (5.4), (5.7), (5.10), and (5.11), the equations of motion (5.3) can be written in the form

$$\dot{x}^i_1 = \frac{1}{1-\zeta_1\zeta_2}\Big((1-\zeta_2)\theta^i_{11} + (1-\zeta_1)\theta^i_{21}\Big), \tag{5.12a}$$

$$\begin{aligned}\dot{p}_{1i} &= -\frac{c^2}{1-\zeta_1\zeta_2}\Big((1-\zeta_2)\eta^i_1 - (1-\zeta_1)\eta^i_2\Big)\\ &\quad - \frac{2c^4}{G}(\Psi_1+\Psi_2-2)n^i_{12}\\ &= \frac{c^2}{1-\zeta_1\zeta_2}\Big((1-\zeta_1)\zeta_2\eta^i_1 - (1-\zeta_2)\zeta_1\eta^i_2\Big)\\ &\quad - c\left(\frac{1}{\Psi_1^7}+\frac{1}{\Psi_2^7}\right)\partial_{1i}(p_{1j}V_{1j}),\end{aligned} \tag{5.12b}$$

where the last equality follows from relations (5.2) and definitions (5.6a)–(5.6b). Similar equations for $\dot{x}^i_2$ and $\dot{p}_{2i}$ hold with the role of labels 1 and 2 exchanged. Knowing the positions and momenta at a given time t, we can solve the algebraic equations (5.2) numerically, calculate η^i_A, θ^i_{AB} and ζ_A by means of Eqs. (5.6), (5.9), (5.11), and determine $x^i_A(t+dt)$ as well as $p_{Ai}(t+dt)$ from the evolution equations (5.12).

VI. LAST STABLE CIRCULAR ORBIT

As an example of application of our skeleton Hamiltonian, we shall discuss in this section the parameters of the last stable (circular) orbit (LSO) in the relative motion of two inspiraling compact objects. This class of systems is of primary importance, being one of the most promising sources of detectable gravitational waves for ground-interferometry experiments as LIGO (Laser Interferometer Gravitational Wave Observatory), VIRGO, GEO600, and TAMA300. Because of gravitational-wave emission, the separation between the bodies decreases adiabatically in time while the frequency increases according to the relativistic version of the Kepler law. The process ends as soon as the first instable orbit is reached before the coalescence phase. The point where this occurs changes according to the model (see Refs. [29, 30] and references therein for some comparisons).

For binary black holes the skeleton Hamiltonian reads $H=(\alpha_1+\alpha_2)c^2$, where α_1 and α_2 are solutions of the coupled system of equations (3.17). It is impossible to solve this system analytically, but it is not difficult to solve it perturbatively within the post-Newtonian setting, treating $\varepsilon \equiv 1/c^2$ as a small parameter. At the nPN approximation, we search the post-Newtonian expansion of the coefficients α_A in the form

$$\alpha_A = m_A + \sum_{i=1}^{n+1}\alpha_A^{(i-1)}\varepsilon^i + \mathcal{O}(\varepsilon^{n+2}), \quad A=1,2. \tag{6.1}$$

Note that we need to carry the series up to the order $n+1$ in ε to get an nPN-accurate dynamics with respect to the exact skeleton evolution (which first differs from the exact Einstein's evolution at the 2PN order). We now substitute (6.1) into Eqs. (3.17) and expand their right-hand sides up to the order $n+1$. By identifying the coefficients of different powers ε^i, $i=1,\ldots,n+1$, we obtain a system of algebraic equations for $\alpha_A^{(i)}$. We display here the first three equations:

$$\frac{\alpha_1^{(0)}}{m_1} = \frac{p_{1i}p_{1i}}{2m_1^2} - \frac{Gm_2}{2r_{12}},$$

$$\begin{aligned}\frac{\alpha_1^{(1)}}{m_1} &= -\frac{(p_{1i}p_{1i})^2}{8m_1^4} + \frac{Gm_2}{2r_{12}}\Bigg(-\frac{5p_{1i}p_{1i}}{2m_1^2} + \frac{7p_{1i}p_{2i}}{2m_1m_2}\\ &\qquad + \frac{n^i_{12}n^j_{12}p_{1i}p_{2j}}{2m_1m_2} - \frac{\alpha_2^{(0)}}{m_2}\Bigg) + \frac{G^2m_2^2}{4r_{12}^2},\end{aligned}$$

$$\begin{aligned}\frac{\alpha_1^{(2)}}{m_1} &= \frac{(p_{1i}p_{1i})^3}{16m_1^6} + \frac{Gm_2}{2r_{12}}\Bigg(\frac{9(p_{1i}p_{1i})^2}{8m_1^4} - \frac{5p_{1i}p_{1i}\alpha_2^{(0)}}{2m_1^2m_2}\\ &\qquad - \frac{\alpha_2^{(1)}}{m_2}\Bigg) + \frac{G^2m_2^2}{2r_{12}^2}\Bigg(\frac{15p_{1i}p_{1i}}{4m_1^2} - \frac{49p_{1i}p_{2i}}{4m_1m_2}\\ &\qquad - \frac{7n^i_{12}n^j_{12}p_{1i}p_{2j}}{4m_1m_2} + \frac{\alpha_2^{(0)}}{m_2}\Bigg) - \frac{G^3m_2^3}{8r_{12}^3};\end{aligned}$$

similar equalities hold with labels 1 and 2 exchanged. It is straightforward to solve up this system to the required order.

For convenience, we shall consider instead of H the dimensionless Hamiltonian (after ruling out the constant rest-mass contribution)

$$\hat{H} \equiv \frac{H-(m_1+m_2)c^2}{\mu c^2},$$

$\mu \equiv m_1 m_2/(m_1+m_2)$ denoting the reduced mass of the system. At the nPN level, we have

$$\hat{H} = \sum_{A=1}^{2} \sum_{i=1}^{n+1} \frac{\alpha_A^{(i-1)}}{\mu} \varepsilon^i + \mathcal{O}\left(\varepsilon^{n+2}\right). \tag{6.2}$$

We restrict our study to circular orbits in the center-of-mass reference frame. Within the ADM gauge, the frame shift is achieved by imposing the condition

$$p_{1i} + p_{2i} = 0\,, \tag{6.3}$$

whereas circularity requires that

$$n_{12}^i p_{1i} = n_{12}^i p_{2i} = 0\,. \tag{6.4}$$

The form of $\hat{H}$ becomes particularly simple when use is made of the reduced relative position and momentum:

$$r^i \equiv \frac{x_1^i - x_2^i}{G(m_1+m_2)}\,, \quad p_i \equiv \frac{p_{1i}}{\mu} = -\frac{p_{2i}}{\mu}\,.$$

When both relations (6.3) and (6.4) hold, the Hamiltonian (6.2) can be expressed with the help of r^i and p_i only. Notice also that it depends on the masses m_1 and m_2 exclusively through the symmetric mass ratio $\nu \equiv m_1 m_2/(m_1+m_2)^2$.

In the next step, we introduce the conserved (reduced) angular momentum j along the cirular orbit of radius r:

$$p^2 = \frac{j^2}{r^2}. \tag{6.5}$$

After eliminating the linear momentum p_i by means of Eq. (6.5), the dimensionless energy $E \equiv \hat{H}(r,j)$ of the binary system becomes a function of the variables r and j:

$$\begin{aligned}\frac{E}{\varepsilon} = \frac{j^2}{2r^2} - \frac{1}{r} + \left(\frac{1}{r^2} - \frac{j^2}{r^3}(3+\nu) + \frac{j^4}{4r^4}(-1+3\nu)\right)\frac{\varepsilon}{2} \\ + \Bigg(-\frac{1}{2r^3}(1+\nu) + \frac{j^2}{r^4}(5+\nu) + \frac{5j^4}{4r^5}(1-4\nu) \\ + \frac{j^6}{8r^6}(1-5\nu+5\nu^2)\Bigg)\frac{\varepsilon^2}{2} + \mathcal{O}(\varepsilon^3).\end{aligned} \tag{6.6}$$

The coordinate radius r of the circular orbit and the angular momentum j are related through one of the canonical equations of motion, namely

$$\frac{\partial \hat{H}(r,j)}{\partial r} = 0\,, \tag{6.7}$$

a relation that may be regarded as expressing the quasi-stationarity of the configuration. Remarkably, it can be shown in this case, i.e., for circular motion, that the ADM mass of the system equates the so-called Komar mass M_{K}, i.e., the mass-like quantity proportional to the monopole part of g_{00} at spatial infinity satisfying the relation $g_{00} = -1 + 4A_3 M_{\mathrm{K}}/r + \mathcal{O}(1/r^2)$ [31]. The proof of the equality $M_{\mathrm{ADM}} = M_{\mathrm{K}}$ is given in Appendix B for an arbitrary number of space dimensions.

Equation (6.7) is solved perturbatively for $r = r(j)$. By plugging the result into Eq. (6.6), we find the gauge invariant link between the energy E and the angular momentum j in the case of circular motion:

$$\begin{aligned}\frac{E}{\varepsilon} =& \frac{\hat{H}(r(j),j)}{\varepsilon} \\ =& -\frac{1}{2j^2} - \frac{1}{8j^4}(9+\nu)\varepsilon \\ & - \frac{1}{16j^6}(81+41\nu-5\nu^2)\varepsilon^2 + \mathcal{O}(\varepsilon^3).\end{aligned} \tag{6.8}$$

It is convenient to rewrite formula (6.8) in terms of the dimensionless variable

$$x \equiv \left(\sqrt{\varepsilon}\,\frac{dE}{dj}\right)^{\frac{2}{3}}. \tag{6.9}$$

The parameter x is now treated as a small quantity of the order ε [cf. Eq. (6.8)]. The definition (6.9) establishes the link between x and j. We invert it order by order to obtain j as a function of x. Inserting $j = j(x)$ into Eq. (6.8) leads to a gauge invariant expression of the energy E depending on the parameter x. The calculation is performed at the 10PN approximation in order to come reasonably close to the exact result. We display here the explicit formula with the 3PN-accuracy. It reads

$$\begin{aligned}E =& -\frac{x}{2} + \left(\frac{3}{8} + \frac{\nu}{24}\right)x^2 + \left(\frac{27}{16} + \frac{29}{16}\nu - \frac{17}{48}\nu^2\right)x^3 \\ &+ \left(\frac{675}{128} + \frac{8585}{384}\nu - \frac{7985}{192}\nu^2 + \frac{1115}{10368}\nu^3\right)x^4 \\ &+ \sum_{i=5}^{11} e_i x^i + \mathcal{O}(x^{12}).\end{aligned} \tag{6.10}$$

The higher order post-Newtonian coefficients e_i (for $i = 5, \ldots, 11$) can be found in Appendix C.

We want to compare the result of Eq. (6.10) with the relation $E_{\mathrm{WM}} = E_{\mathrm{WM}}(x)$ of the Wilson-Mathews model (remember that in this model the post-Newtonian expansion terminates at the 2PN order)[2]:

$$\begin{aligned}E_{\mathrm{WM}} =& -\frac{x}{2} + \left(\frac{3}{8} + \frac{\nu}{24}\right)x^2 \\ &+ \left(\frac{27}{16} - \frac{39}{16}\nu - \frac{17}{48}\nu^2\right)x^3 + \mathcal{O}(x^4).\end{aligned} \tag{6.11}$$

The 2PN coefficient proportional to ν^2 differs from that of the post-Newtonian computations carried out in general relativity. Indeed, the energy for circular orbits in

---

[2] This relation is obtained by taking $n = j$ in Eq. (C4) of Ref. [32] and replacing there the angular momentum j by $j_{\mathrm{WM}}(x)$ according to Eq. (B1) of Ref. [29].

Einsteinian theory, up to the 3PN order, is [29]

$$E_{\rm GR} = -\frac{x}{2} + \left(\frac{3}{8} + \frac{\nu}{24}\right)x^2 + \left(\frac{27}{16} - \frac{19}{16}\nu + \frac{1}{48}\nu^2\right)x^3 + \left(\frac{675}{128} + \left(\frac{205}{192}\pi^2 - \frac{34445}{1152}\right)\nu + \frac{155}{192}\nu^2 + \frac{35}{10368}\nu^3\right)x^4 + \mathcal{O}(x^5), \qquad (6.12)$$

where, according to Ref. [7] we set the parameter $\omega_{\rm static}$ (defined in paper [33]) equal to zero ($\omega_{\rm static} = 0$), getting the solution corresponding to the Brill-Lindquist initial data.[3] By comparing Eqs. (6.10) and (6.12) it is obvious that our energy E starts to differ from the general-relativistic one, $E_{\rm GR}$, at the 2PN order.

The position $x_{\rm LSO}$ of the last stable orbit is determined from the requirement that $(dE/dx)_{x=x_{\rm LSO}} = 0$.[4] The corresponding energy $E_{\rm LSO}$ [computed by means of Eq. (6.10)] is depicted in Fig. 1 for equal-mass binaries (i.e., for $\nu = 1/4$) as a function of the dimensionless orbital frequency $x^{3/2}_{\rm LSO}$. Numerical values are given in Table I. The case of a test particle moving in a Schwarzschild background is also plotted for comparison. Similarly to what happens in Einsteinian theory [30], the instabilities arise "earlier" (i.e., at lower frequency) for higher post-Newtonian orders in the skeleton model. The 2PN last stable orbit of our skeleton dynamics is quite far from that of general relativity in the sense that the absolute difference $\delta(x^{3/2}_{\rm LSO})_{\rm 2PN}$ $(\delta(E_{\rm LSO})_{\rm 2PN})$ between the skeleton frequency $(x^{3/2}_{\rm LSO})_{\rm 2PN} = 0.1077$ (the skeleton energy $(E_{\rm LSO})_{\rm 2PN} = -0.0689$) and the Einsteinian one $(x^{3/2}_{\substack{\rm GR\\ \rm LSO}})_{\rm 2PN} = 0.1371$ $((E_{\substack{\rm GR\\ \rm LSO}})_{\rm 2PN} = -0.0795)$ is not negligible with respect to the general-relativistic value: $\delta(x^{3/2}_{\rm LSO})_{\rm 2PN}/(x^{3/2}_{\substack{\rm GR\\ \rm LSO}})_{\rm 2PN} = 21\%$ $(\delta(E_{\rm LSO})_{\rm 2PN}/(E_{\substack{\rm GR\\ \rm LSO}})_{\rm 2PN} = 13\%)$. The location of the Wilson-Mathews data suggests that modifications of the extrinsic curvature affect the parameters of the last stable orbit more than the choice of a conformally flat space metric [43]. The 3PN skeleton values are even further away from the Einsteinian 3PN ones, but they are closer to those of the effective one-body approach improved by Padé approximants than the straight 3PN Einsteinian values. The 10PN data are particularly interesting, considering that they may be regarded as good approximants to the full skeleton model. Remarkably, their position in the $(x^{3/2}, E)$ plane relative to the 3PN data is nearly the same as that of the point referring to numerical simulations relative to the point referring to the 3PN data in general relativity.

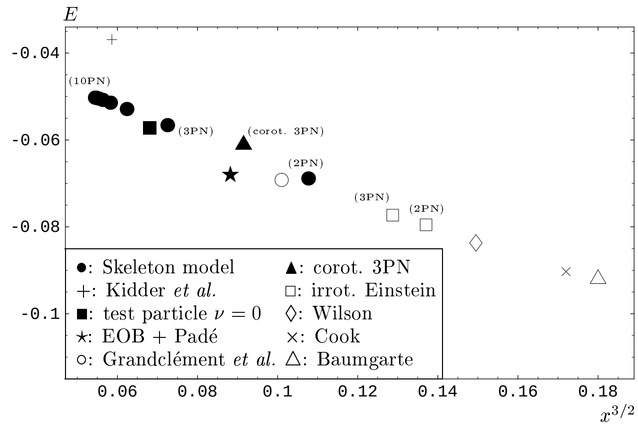

FIG. 1: Position of the last stable circular orbit in the frequency/energy plane for equal-mass binaries in different models. The sequence of black circles corresponds to different post-Newtonian truncations of the skeleton energy (from the 2PN to the 10PN order when moving to the left). The values of the 3PN Einsteinian energy for non-spinning black holes are computed in Ref. [30] based on the works of Damour, Jaranowski, Schäfer [7, 35, 36, 37] and Blanchet, Faye [38, 39, 40] with $\omega_{\rm static} = 0$. The modifications coming from the spins in the corotational case are given in Ref. [30]. The plus cross refers to the point obtained within the hybrid method of Kidder, Will, and Wiseman [41], in which terms are added to the 2PN equations of motion in order to get the exact test-particle limit. The data corresponding to the white circle have been calculated from a sequence of quasi-equilibrium configurations for corotating Misner black holes by Grandclément and collaborators [2, 3] using a Wilson-Mathews-type approximation (conformal thin-sandwich approach [4]). The time cross indicates the values computed by Cook [25] for two Misner black holes assuming a Bowen-York-like extrinsic curvature (conformal transverse-traceless approach [4]). The white triangle shows the position of the last stable circular orbit determined by Baumgarte [26] with the help of the puncture method, which means in particular that the black holes are of Brill-Lindquist type, and that the extrinsic curvature is the one of Bowen-York (conformal transverse-traceless approach). The star refers to the effective-one-body approach with Padé approximant [29]. Finally, we have indicated the frequency and the energy in the test-body limit ($\nu = 0$).

In Fig. 1, the relative location of the last stable orbit indicated by the white circle obtained from numerical computations with respect to the 3PN result for corotating bodies (black triangle) is somehow plausible if we believe that it is essentially due to the Wilson-Mathews truncation. It is indeed comparable to the relative location of irrotational 2PN Einsteinian and Wilson-Mathews data. We also want to point out that a realistic estimation of the last stable orbit does not necessarily follow

[3] If the space transformation which relates the Brill-Lindquist solution with the Misner-Lindquist one through 3PN order [34] is regarded as induced by canonical transformation, the Brill-Lindquist and Misner-Lindquist solutions are physically identical at this level.

[4] This determination is not unique for approximate expressions. The method presented in paper [42] is also valuable. It will not be employed here since we are interested in the comparison with other works.

TABLE I: The last stable circular orbit parameters in the skeleton model truncated at various PN orders for equal-mass binaries.

	1PN	2PN	3PN	4PN	5PN	6PN	7PN	8PN	9PN	10PN
$x^{3/2}$	0.5224	0.1077	0.0725	0.0624	0.0583	0.0563	0.0554	0.0548	0.0546	0.0544
E	−0.1622	−0.0689	−0.0566	−0.0529	−0.0514	−0.0508	−0.0505	−0.0503	−0.0503	−0.0502

from the standard 3PN calculation. It may well be close to the star-like point derived from the effective one-body model with Padé approximant. There is currently no definite argument to decide what is the most likely statement. As a result, we regard the position of these points as defining the present uncertainty in the location of the last stable orbit derived analytically.

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APPENDIX A: POLYNOMIAL EQUATIONS FOR THE COEFFICIENTS Ψ_A IN $d = 3$ DIMENSIONS

Equation (5.2) for $A = 2$, after multiplication by Ψ_1^7, can be written in the form $\Psi_2\Psi_1^7 = P_{1\,(7)} + \Psi_1^4 P_{1\,(4)}^{1/2}$, where $P_{1\,(7)} \equiv \Psi_1^7 + Gp_{1i}V_{1i}/(2r_{12}c^3)$ and $P_{1\,(4)} \equiv [Gm_1/(2r_{12}c^2)]^2[\Psi_1^4 + p_1^2/(m_1^2c^2)]$ are polynomials in Ψ_1 with non-zero constant terms, of order 7 and 4, respectively. A similar relation holds for $A = 1$. We now subtract $P_{2\,(7)}$ from the left and right-hand side, and square the new identity. We obtain:

$$(\Psi_1-1)^2\Psi_2^{14} - \frac{Gp_{2i}V_{2i}}{r_{12}c^3}(\Psi_1-1)\Psi_2^7 + \left(\frac{Gp_{2i}V_{2i}}{2r_{12}c^3}\right)^2 = \left(\frac{Gm_2}{2r_{12}c^2}\right)^2\Psi_2^8\left(\Psi_2^4+\frac{p_2^2}{m_2^2c^2}\right). \quad \text{(A1)}$$

Next, we multiply the equality (A1) by $(\Psi_1^7)^{14} = \Psi_1^{98}$, make the substitution $\Psi_2 \to \Psi_1^{-7}(P_{1\,(7)} + \Psi_1^4 P_{1\,(4)}^{1/2})$, and expand the terms $(P_{1\,(7)} + \Psi_1^4 P_{1\,(4)}^{1/2})^n$, $n \in \mathbb{N}^*$, in powers of $P_{1\,(4)}^{1/2}$ with the help of the binomial formula. If n is even, the contribution $M_1^{(n)} \equiv \sum_{k=0}^{n/2}\binom{n}{2k}P_{1\,(7)}^{2k}\Psi_1^{8(n/2-k)}P_{1\,(4)}^{n/2-k}$, with $\binom{n}{2k} = n!/[(2k)!(n-2k)!]$, is of polynomial kind, whereas that of $\sum_{k=0}^{n/2-1}\binom{n}{2k+1}P_{(7)}^{2k+1}\Psi_1^{8(n/2-k-1)+4}P_{1\,(4)}^{n/2-k-1+1/2}$ has the form of a polynomial $N_1^{(n)}$ multiplied by the algebraic function $P_{1\,(4)}^{1/2}$; $M_1^{(n)}$ and $N_1^{(n)}$ have degrees $\max_{0\le k\le n/2}[7\times 2k + 12\times(n/2-k)] = 7n$ and $\max_{0\le k\le n/2-1}[7\times(2k+1)+12\times(n/2-k-1)+4] = 7n-3$, respectively. The same decomposition holds for odd n with different expressions for M_1 and N_1. At last, we gather all polynomial quantities into the first member of the equality while the non polynomial contributions $P_{1\,(4)}^{1/2}$ are recast to the second member. The final equation is

$$\begin{aligned}
&(\Psi_1-1)^2\sum_{k=0}^{7}\binom{14}{2k}P_{1\,(7)}^{2k}\Psi_1^{8(7-k)}P_{1\,(4)}^{7-k} - \frac{Gp_{2i}V_{2i}}{r_{12}c^3}(\Psi_1-1)\Psi_1^{49}\sum_{k=0}^{3}\binom{7}{2k+1}P_{1\,(7)}^{2k+1}\Psi_1^{8(3-k)}P_{1\,(4)}^{3-k} + \left(\frac{Gp_{2i}V_{2i}}{2r_{12}c^3}\right)^2\Psi_1^{98}\\
&-\left(\frac{Gm_2\Psi_1^7}{2r_{12}c^2}\right)^2\left[\sum_{k=0}^{6}\binom{12}{2k}P_{1\,(7)}^{2k}\Psi_1^{8(6-k)}P_{1\,(4)}^{6-k} + \frac{p_2^2\Psi_1^{28}}{m_2^2c^2}\sum_{k=0}^{4}\binom{8}{2k}P_{1\,(7)}^{2k}\Psi_1^{8(4-k)}P_{1\,(4)}^{4-k}\right]\\
&= -P_{1\,(4)}^{1/2}\left[(\Psi_1-1)^2\sum_{k=0}^{6}\binom{14}{2k+1}P_{1\,(7)}^{2k+1}\Psi_1^{4(13-2k)}P_{1\,(4)}^{6-k} - \frac{Gp_{2i}V_{2i}}{r_{12}c^3}(\Psi_1-1)\Psi_1^{49}\sum_{k=0}^{3}\binom{7}{2k}P_{1\,(7)}^{2k}\Psi_1^{4(7-2k)}P_{1\,(4)}^{3-k}\right.\\
&\qquad\left.-\left(\frac{Gm_2\Psi_1^7}{2r_{12}c^2}\right)^2\left(\sum_{k=0}^{5}\binom{12}{2k+1}P_{1\,(7)}^{2k+1}\Psi_1^{4(11-2k)}P_{1\,(4)}^{5-k} + \frac{p_2^2\Psi_1^{28}}{m_2^2c^2}\sum_{k=0}^{3}\binom{8}{2k+1}P_{1\,(7)}^{2k+1}\Psi_1^{4(7-2k)}P_{1\,(4)}^{3-k}\right)\right]. \quad \text{(A2)}
\end{aligned}$$

We see that the left-hand side of Eq. (A2) is made of a sum of polynomials of degrees 100, 99, 98, 98, 98, the first of them involving the only non-zero constant term. The right-hand side is the product of $P_{1\,(4)}^{1/2}$ and polynomials of degrees 97, 96, 95, 95 respectively that cancel when $\Psi_1 = 0$. As a consequence, Ψ_1 is the root of a polynomial of degree 200 with a non-zero constant term.

APPENDIX B: PROOF OF EQUALITY OF ADM AND KOMAR MASSES IN THE CASE OF TWO BODIES ALONG CIRCULAR ORBITS

In this appendix, we restrict ourselves to a binary system. The black holes are thus labeled by $A, B, \ldots \in \{1, 2\}$. Our starting point is the system of two sets of equations satisfied by the monopoles α_A and β_A entering the conformal factor Ψ and the auxiliary function $\chi \equiv N\Psi$:

$$\alpha_A = \frac{p_{Ai} V_{Ai}}{c} \Psi_A^{-\frac{3d-2}{d-2}} + m_A \Psi_A^{-1} \left(1 + \frac{p_A^2}{m_A^2 c^2} \Psi_A^{-\frac{4}{d-2}}\right)^{\frac{1}{2}}, \qquad \text{(B1a)}$$

$$\beta_A = \chi_A \left(\frac{3d-2}{d-2} \frac{p_{Ai} V_{Ai}}{c} \Psi_A^{-\frac{4(d-1)}{d-2}} + m_A \Psi_A^{-2} \left(1 + \frac{p_A^2}{m_A^2 c^2} \Psi_A^{-\frac{4}{d-2}}\right)^{-\frac{1}{2}} \times \left(1 + \frac{d}{d-2} \frac{p_A^2}{m_A^2 c^2} \Psi_A^{-\frac{4}{d-2}}\right) \right), \qquad \text{(B1b)}$$

with (here $B \neq A$)

$$\Psi_A = 1 + A_d \frac{\alpha_B}{r_{12}^{d-2}}, \qquad \text{(B2a)}$$

$$\chi_A = 1 - A_d \frac{\beta_B}{r_{12}^{d-2}}. \qquad \text{(B2b)}$$

It is straightforward to show that for a binary system described by the functions Ψ and χ of the purely monopolar form given by Eqs. (3.14) and (4.8), the ADM mass coincides with its Komar mass if and only if the quantities α_A and β_A satisfy the simple relation:

$$\sum_{A=1}^{2} (\alpha_A - \beta_A) = 0. \qquad \text{(B3)}$$

We shall assume that the relative orbits are circular. After introducing the reduced relative momentum p_i and eliminating its square through the relation $p^2 = G^2(m_1 + m_2)^2 j^2 / r_{12}^2$, all quantities become functions of r_{12} and j only. By virtue of Eqs. (B1a), the effective mass α_A can then be treated as a function of the *three* variables Ψ_A, r_{12}, and j:

$$\alpha_A = \alpha_A(\Psi_A(r_{12}, j), r_{12}, j). \qquad \text{(B4)}$$

With the help of Eqs. (B1), it is not difficult to check that the following relations, crucial for our proof, are fulfilled by α_A:

$$\frac{\partial \alpha_A(\Psi_A, r_{12}, j)}{\partial \Psi_A} = -\frac{\beta_A}{\chi_A}, \qquad \text{(B5a)}$$

$$\frac{\partial \alpha_A(\Psi_A, r_{12}, j)}{\partial r_{12}} = \frac{d-2}{2r_{12}} \left(\alpha_A - \beta_A \frac{\Psi_A}{\chi_A}\right). \qquad \text{(B5b)}$$

Note that, writing down Eqs. (B5b), we have already employed the relations (B5a).

The ADM Hamiltonian, restricted to circular orbits, reads

$$H(r_{12}, j) = c^2 \sum_{A=1}^{2} \alpha_A(r_{12}, j). \qquad \text{(B6)}$$

The circularity condition implies $\partial H(r_{12}, j)/\partial r_{12} = 0$, which is equivalent to

$$\sum_{A=1}^{2} \frac{\partial \alpha_A(r_{12}, j)}{\partial r_{12}} = 0. \qquad \text{(B7)}$$

After substituting (B6) into (B7) we obtain the equality (valid along circular orbits):

$$\sum_{A=1}^{2} \left(\frac{\partial \alpha_A}{\partial \Psi_A} \frac{\partial \Psi_A}{\partial r_{12}} + \frac{\partial \alpha_A}{\partial r_{12}} \right) = 0.$$

According to Eq. (B5), it can be rewritten as

$$\sum_{A=1}^{2} \left(\frac{\beta_A}{\chi_A} \frac{\partial \Psi_A}{\partial r_{12}} + \frac{d-2}{2r_{12}} \left(\beta_A \frac{\Psi_A}{\chi_A} - \alpha_A \right) \right) = 0. \qquad \text{(B8)}$$

On the other hand, the equations (B2a) in the circular case take the form

$$\Psi_A(r_{12}, j) = 1 + A_d \frac{\alpha_B(\Psi_B(r_{12}, j), r_{12}, j)}{r_{12}^{d-2}}, \quad B \neq A.$$

By differentiating both sides with respect to r_{12} and resorting to relations (B5), we obtain the system of two equations (where $B \neq A$)

$$\frac{\partial \Psi_A}{\partial r_{12}} = -\frac{d-2}{2 r_{12}^{d-1}} A_d \left(\alpha_B + \beta_B \frac{\Psi_B}{\chi_B} \right) - \frac{A_d}{r_{12}^{d-2}} \frac{\beta_B}{\chi_B} \frac{\partial \Psi_B}{\partial r_{12}},$$

which we solve with respect to the derivatives $\partial \Psi_A / \partial r_{12}$. After eliminating Ψ_A and χ_A by means of relations (B2), the solution reduces to

$$\frac{\partial \Psi_A}{\partial r_{12}} = -\frac{d-2}{2 r_{12}^{d-1}} A_d (\alpha_B + \beta_B), \quad B \neq A. \qquad \text{(B9)}$$

To complete the proof of Eq. (B3), it is enough to substitute Eqs. (B9) into the left-hand side of equality (B8) and make use of Eqs. (B2).

APPENDIX C: ENERGY ALONG CIRULAR ORBITS AS A FUNCTION OF THE PARAMETER x UP TO 10PN ORDER

The coefficients e_i (for $i = 5, \ldots, 11$) entering formula (6.10) for energy E along cirular orbits as a function of the parameter x read

$$
\begin{aligned}
e_5 &= \frac{3969}{256} + \frac{105553}{768}\nu - \frac{799673}{2304}\nu^2 + \frac{502145}{3456}\nu^3 + \frac{135247}{62208}\nu^4, \\
e_6 &= \frac{45927}{1024} + \frac{662619}{1024}\nu - \frac{233737}{128}\nu^2 + \frac{1162513}{1024}\nu^3 + \frac{56481}{1024}\nu^4 - \frac{3777}{1024}\nu^5, \\
e_7 &= \frac{264627}{2048} + \frac{16343855}{6144}\nu - \frac{136513223}{18432}\nu^2 + \frac{67182599}{27648}\nu^3 + \frac{249565613}{31104}\nu^4 - \frac{766814851}{497664}\nu^5 - \frac{148888223}{13436928}\nu^6, \\
e_8 &= \frac{12196899}{32768} + \frac{993480761}{98304}\nu - \frac{403684801}{16384}\nu^2 - \frac{19713422131}{884736}\nu^3 + \frac{532659868169}{3981312}\nu^4 - \frac{1096304004055}{11943936}\nu^5 \\
&\quad + \frac{42034682027}{17915904}\nu^6 + \frac{29197027769}{644972544}\nu^7, \\
e_9 &= \frac{70366725}{65536} + \frac{2394150375}{65536}\nu - \frac{13003419935}{196608}\nu^2 - \frac{9560242585}{32768}\nu^3 + \frac{248408863105}{196608}\nu^4 - \frac{129128080475}{98304}\nu^5 \\
&\quad + \frac{30481497595}{98304}\nu^6 + \frac{189766295}{16384}\nu^7 + \frac{1599565}{65536}\nu^8, \\
e_{10} &= \frac{813439341}{262144} + \frac{100457157571}{786432}\nu - \frac{69694465361}{589824}\nu^2 - \frac{44773413415657}{21233664}\nu^3 + \frac{558727148465215}{63700992}\nu^4 \\
&\quad - \frac{2097054471026245}{191102976}\nu^5 + \frac{500159286898463}{143327232}\nu^6 + \frac{858674754130903}{2579890176}\nu^7 - \frac{233795515077109}{5159780352}\nu^8 \\
&\quad - \frac{177721652120353}{417942208512}\nu^9, \\
e_{11} &= \frac{4710988269}{524288} + \frac{228788885795}{524288}\nu + \frac{411952929451}{4718592}\nu^2 - \frac{28395688669829}{2359296}\nu^3 + \frac{3163918218071455}{63700992}\nu^4 \\
&\quad - \frac{8137959523310875}{127401984}\nu^5 + \frac{6129421057080619}{1146617856}\nu^6 + \frac{9328338011915891}{322486272}\nu^7 - \frac{180325112745666065}{30958682112}\nu^8 \\
&\quad - \frac{117289128020845 85}{278628139008}\nu^9 + \frac{1098645689138995}{2507653251072}\nu^{10}.
\end{aligned}
$$

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